



TITLE:

The Hypergeometric Series and Trigonometric Sums (Coefficient Inequalities in Univalent Function Theory and Related Topics)

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The Hypergeometric Series and Trigonometric Sums

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1. Introduction

Let us consider the following Gauss hypergeometric series,

$$F(-n, 2m; 2m + n + 1; z) = \sum_{k=0}^n \frac{(2m)_k (-n)_k z^k}{(2m + n + 1)_k k!},$$

where m is a positive constant and n is an positive integer. Here $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ denotes the Pochhammer symbol. Consider the unit circle $C: z = e^{it}$ ($0 \leq t \leq 2\pi$). Let

$$R(-n, 2m; 2m + n + 1; t) = \Re\{F(-n, 2m; 2m + n + 1; e^{it})\}.$$

We have

$$R(-n, 2m; 2m + n + 1; t) = \sum_{k=0}^n \frac{(-n)_k (2m)_k}{(2m + n + 1)_k} \frac{\cos kt}{k!}. \quad (1)$$

Let

$$I(-n, 2m; 2m + n + 1; t) = \Im\{F(-n, 2m; 2m + n + 1; e^{it})\}.$$

We have

$$I(-n, 2m; 2m + n + 1; t) = \sum_{k=1}^n \frac{(-n)_k (2m)_k}{(2m + n + 1)_k} \frac{\sin kt}{k!}. \quad (2)$$

From the Gauss hypergeometric series $F(-n, 2m; 2m + n + 1; e^{it})$ we can deduce some formulas of trigonometric sums. These results are listed in the following form.

Formula 1. We obtain

$$\begin{aligned} & |F(-n, 2m; 2m + n + 1; e^{it})|^2 \\ &= \frac{(n+1)_n}{(2m+n+1)_n} \sum_{s=0}^n \frac{(-n)_s (n+1-s)_{n-s} (2m)_s y^s}{(-2n)_s (2m+n+1)_{n-s} s!} \\ &= \sum_{s=0}^n \frac{(2m)_s}{(2m+n+1)_n (2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! y^s, \quad (3) \end{aligned}$$

where $y = 2(1 - \cos t)$.

Formula 2. We obtain

$$\begin{aligned}
 & I(-n+1, 2m; 2m+n+1; t)R(-n, 2m; 2m+n+1; t) \\
 & - R(-n+1, 2m; 2m+n+1; t)I(-n, 2m; 2m+n+1; t) \\
 = & B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \cdots \\
 & + B_{n-2} \sin(n-2)t + B_{n-1} \sin(n-1)t + B_n \sin nt,
 \end{aligned} \tag{4}$$

where B_{n-j} are given by the following formula,

$$\begin{aligned}
 B_{n-r} = & (-1)^{n-r-1} \sum_{k=0}^r \frac{(2m)_{k+n-r} (2m)_k}{(2m+n+1)_{k+n-r} (2m+n+1)_k} \\
 & \cdot \frac{(n-1)!n!(n-r)}{(n+k-r)!(r-k)!k!(n-k)!}, \quad (r = 0, 1, 2, \dots, n-1).
 \end{aligned} \tag{5}$$

Formula 3. We obtain

$$\begin{aligned}
 & n\{I(-n+1, 2m; 2m+n+1; t)R(-n, 2m; 2m+n+1; t) \\
 & - R(-n+1, 2m; 2m+n+1; t)I(-n, 2m; 2m+n+1; t)\} \\
 = & \frac{(n+1)_n}{(2m+n+1)_n} \sum_{s=1}^n \frac{(-n)_s (n+1-s)_{n-s} (2m)_s s y^{s-1}}{(-2n)_s (2m+n+1)_{n-s} s!} \sin t \\
 = & \sum_{s=1}^n \frac{(2m)_s}{(2m+n+1)_n (2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! s y^{s-1} \\
 & \cdot \sin t.
 \end{aligned} \tag{6}$$

Formula 4. We obtain the following combinatorial equality.

$$\begin{aligned}
 & \frac{(2m)_{n-r+1}}{(2m+n+1)_n (2m+n+1)_{r-1}} \binom{n-1}{n-r} \binom{n+r-1}{n} (2(r-1))! \\
 = & \sum_{k=0}^{r-1} \sum_{j=0}^k (-1)^{r+k+1} \frac{(2m)_{j+n-k} (2m)_j}{(2m+n+1)_{j+n-k} (2m+n+1)_j} \\
 & \cdot \frac{(n-1)!n!(n-k)}{(n+j-k)!(k-j)!j!(n-j)!} \binom{2n-r-k}{r-k-1}.
 \end{aligned} \tag{7}$$

2. On Formula 1

If m is a positive integer it is known in [5]. In this report m is assumed to be a positive constant. The curve of the function $F(-n, 2m; 2m + n + 1; e^{it})$ of the variable t is symmetric with respect to the real axis in the complex plane. We make use of Watson's formula

$$\begin{aligned} & F(-n, b; c; z) F(-n, b; c; Z) \\ &= \frac{(c-b)_n}{(c)_n} F_4[-n, b; c, 1-n+b-c; zZ, (1-z)(1-Z)] \end{aligned}$$

in Slater's book (cf. [3. (8.4.2)]). Let $b = 2m$, $c = 2m + n + 1$. Then

$$\begin{aligned} & F_4[-n, 2m; 2m + n + 1, -2n; zZ, (1-z)(1-Z)] \\ &= \sum_{r=0}^n \sum_{s=0}^n \frac{(-n)_{r+s} (2m)_{r+s} (zZ)^r [(1-z)(1-Z)]^s}{(2m+n+1)_r (-2n)_s r! s!}. \end{aligned}$$

Let $z = e^{it}$, $Z = e^{-it}$. Then $zZ = 1$ and $(1-z)(1-Z) = 2(1 - \cos t)$. Let $x = 1$ and $y = 2(1 - \cos t)$. From the above F_4 we obtain

$$\begin{aligned} & |F(-n, 2m; 2m + n + 1; e^{it})|^2 \\ &= \frac{(n+1)_n}{(2m+n+1)_n} \sum_{r=0}^n \sum_{s=0}^n \frac{(-n)_{r+s} (2m)_{r+s} x^r y^s}{(2m+n+1)_r (-2n)_s r! s!}. \end{aligned} \quad (8)$$

It holds that

$$\begin{aligned} & |F(-n, 2m; 2m + n + 1; e^{it})|^2 \\ &= \frac{(n+1)_n}{(2m+n+1)_n} \sum_{s=0}^n \frac{(-n)_s (2m)_s y^s}{(-2n)_s s!} \left\{ \sum_{r=0}^{n-s} \frac{(-n+s)_r (2m+s)_r x^r}{(2m+n+1)_r r!} \right\}. \end{aligned}$$

We see that the right hand side of the above equality is strictly positive for all t in the interval $[0, \pi]$ since it holds that

$$\begin{aligned} & \sum_{r=0}^{n-s} \frac{(-n+s)_r (2m+s)_r x^r}{(2m+n+1)_r r!} \\ &= \frac{\Gamma(2m+n+1)}{\Gamma(2m+s)\Gamma(n-s+1)} \int_0^1 \tau^{2m+s-1} (1-\tau)^{n-s} (1-x\tau)^{n-s} d\tau. \end{aligned}$$

When $x = 1$ we see by Vandermonde's formula that

$$\begin{aligned} & |F(-n, 2m; 2m + n + 1; e^{it})|^2 \\ &= \frac{(n+1)_n}{(2m+n+1)_n} \sum_{s=0}^n \frac{(-n)_s (n+1-s)_{n-s} (2m)_s y^s}{(-2n)_s (2m+n+1)_{n-s} s!} \\ &= \sum_{s=0}^n \frac{(2m)_s}{(2m+n+1)_n (2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! y^s, \end{aligned}$$

where $y = 2(1 - \cos t)$.

3. On Formula 2

From the following relation

$$aF(a+1, b; c; z) = aF(a, b; c; z) + z \frac{d}{dz} F(a, b; c; z)$$

we have

$$\begin{aligned} & \frac{d}{dt} F(-n, 2m; 2m+n+1; e^{it}) \\ &= \frac{d}{dz} F(-n, 2m; 2m+n+1; z) i e^{it} \\ &= \{F(-n+1, 2m; 2m+n+1; e^{it}) - F(-n, 2m; 2m+n+1; e^{it})\}(-n)i \end{aligned}$$

and we see that

$$\begin{aligned} & \frac{d}{dt} |F(-n, 2m; 2m+n+1; e^{it})|^2 \\ &= \left\{ \frac{d}{dt} F(-n, 2m; 2m+n+1; e^{it}) \right\} F(-n, 2m; 2m+n+1; e^{-it}) \\ & \quad + F(-n, 2m; 2m+n+1; e^{it}) \left\{ \frac{d}{dt} F(-n, 2m; 2m+n+1; e^{-it}) \right\} \\ &= \left[\{F(-n+1, 2m; 2m+n+1; e^{it}) - F(-n, 2m; 2m+n+1; e^{it})\} \right. \\ & \quad \cdot (-n)i F(-n, 2m; 2m+n+1; e^{-it}) \\ & \quad + F(-n, 2m; 2m+n+1; e^{it}) \\ & \quad \cdot \{F(-n+1, 2m; 2m+n+1; e^{-it}) - F(-n, 2m; 2m+n+1; e^{-it})\}ni \Big] \\ &= \left[F(-n+1, 2m; 2m+n+1; e^{it}) F(-n, 2m; 2m+n+1; e^{-it}) (-n)i \right. \\ & \quad + F(-n, 2m; 2m+n+1; e^{it}) F(-n+1, 2m; 2m+n+1; e^{-it}) ni \Big] \\ &= 2\Re \left[(-in) F(-n+1, 2m; 2m+n+1; e^{it}) \right. \\ & \quad \cdot F(-n, 2m; 2m+n+1; e^{-it}) \Big] \\ &= 2(-in)i \left\{ -I(-n, 2m; 2m+n+1; t) R(-n+1, 2m; 2m+n+1; t) \right. \\ & \quad + R(-n, 2m; 2m+n+1; t) I(-n+1, 2m; 2m+n+1; t) \Big\} \\ &= 2n \left\{ I(-n+1, 2m; 2m+n+1; t) R(-n, 2m; 2m+n+1; t) \right. \\ & \quad \left. - R(-n+1, 2m; 2m+n+1; t) I(-n, 2m; 2m+n+1; t) \right\}. \tag{9} \end{aligned}$$

Let us denote the derivative of $|F(-n, 2m; 2m+n+1; e^{it})|^2$ by $2d(t)$. We obtain

$$d(t) = n \left\{ I(-n+1, 2m; 2m+n+1; t) R(-n, 2m; 2m+n+1; t) \right.$$

$$-R(-n+1, 2m; 2m+n+1; t)I(-n, 2m; 2m+n+1; t)\}. (10)$$

We will show Formula 2. We have

$$\begin{aligned} & d(t) \\ &= \Re[(-in)F(-n+1, 2m; 2m+n+1; e^{it})F(-n, 2m; 2m+n+1; e^{-it})]. \end{aligned} \quad (11)$$

We see that

$$\begin{aligned} & F(-n+1, 2m; 2m+n+1; e^{it})F(-n, 2m; 2m+n+1; e^{-it}) \\ &= \left\{ \sum_{k=0}^{n-1} \frac{(2m)_k (-n+1)_k e^{ikt}}{(2m+n+1)_k k!} \right\} \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j e^{-ijt}}{(2m+n+1)_j j!} \right\} \\ &= \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j e^{-ijt}}{(2m+n+1)_j j!} \right\} \\ &+ \frac{(2m)_1 (-n+1)_1}{(2m+n+1)_1 1!} \cdot \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j e^{-i(j-1)t}}{(2m+n+1)_j j!} \right\} \\ &+ \frac{(2m)_2 (-n+1)_2}{(2m+n+1)_2 2!} \cdot \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j e^{-i(j-2)t}}{(2m+n+1)_j j!} \right\} \\ &+ \dots \\ &+ \frac{(2m)_{n-1} (-n+1)_{n-1}}{(2m+n+1)_{n-1} (n-1)!} \cdot \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j e^{-i(j-n+1)t}}{(2m+n+1)_j j!} \right\}. \end{aligned} \quad (12)$$

From the above we obtain

$$\begin{aligned} & \Re\{i F(-n+1, 2m; 2m+n+1; e^{it})F(-n, 2m; 2m+n+1; e^{-it})\} \\ &= \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j \sin(jt)}{(2m+n+1)_j j!} \right\} \\ &+ \frac{(2m)_1 (-n+1)_1}{(2m+n+1)_1 1!} \cdot \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j \sin(j-1)t}{(2m+n+1)_j j!} \right\} \\ &+ \frac{(2m)_2 (-n+1)_2}{(2m+n+1)_2 2!} \cdot \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j \sin(j-2)t}{(2m+n+1)_j j!} \right\} \\ &+ \dots \\ &+ \frac{(2m)_{n-1} (-n+1)_{n-1}}{(2m+n+1)_{n-1} (n-1)!} \cdot \left\{ \sum_{j=0}^n \frac{(2m)_j (-n)_j \sin(j-n+1)t}{(2m+n+1)_j j!} \right\} \end{aligned} \quad (13)$$

Summing the terms up with respect to the same sine functions in the above we obtain

$$\Re\{i F(-n+1, 2m; 2m+n+1; e^{it})F(-n, 2m; 2m+n+1; e^{-it})\}$$

$$\begin{aligned}
&= \left\{ \sum_{k=0}^{n-1} \frac{(2m)_k (-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+1} (-n)_{k+1}}{(2m+n+1)_{k+1} (k+1)!} \right\} \sin t \\
&+ \left\{ \sum_{k=0}^{n-2} \frac{(2m)_k (-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+2} (-n)_{k+2}}{(2m+n+1)_{k+2} (k+2)!} \right\} \sin 2t \\
&+ \left\{ \sum_{k=0}^{n-3} \frac{(2m)_k (-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+3} (-n)_{k+3}}{(2m+n+1)_{k+3} (k+3)!} \right\} \sin 3t \\
&+ \dots \\
&+ \left\{ \sum_{k=0}^3 \frac{(2m)_k (-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-3} (-n)_{k+n-3}}{(2m+n+1)_{k+n-3} (k+n-3)!} \right\} \\
&\quad \cdot \sin(n-3)t \\
&+ \left\{ \sum_{k=0}^2 \frac{(2m)_k (-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-2} (-n)_{k+n-2}}{(2m+n+1)_{k+n-2} (k+n-2)!} \right\} \\
&\quad \cdot \sin(n-2)t \\
&+ \left\{ \sum_{k=0}^1 \frac{(2m)_k (-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-1} (-n)_{k+n-1}}{(2m+n+1)_{k+n-1} (k+n-1)!} \right\} \\
&\quad \cdot \sin(n-1)t \\
&+ \frac{(2m)_n (-n)_n}{(2m+n+1)_n n!} \sin nt \\
&- \left\{ \sum_{j=0}^{n-2} \frac{(2m)_{j+1} (-n+1)_{j+1}}{(2m+n+1)_j (j+1)!} \frac{(2m)_j (-n)_j}{(2m+n+1)_j j!} \right\} \sin t \\
&- \left\{ \sum_{j=0}^{n-3} \frac{(2m)_{j+2} (-n+1)_{j+2}}{(2m+n+1)_j k!} \frac{(2m)_j (-n)_j}{(2m+n+1)_j j!} \right\} \sin 2t \\
&- \left\{ \sum_{j=0}^{n-4} \frac{(2m)_{j+3} (-n+1)_{j+3}}{(2m+n+1)_{j+3} (j+3)!} \frac{(2m)_j (-n)_j}{(2m+n+1)_j j!} \right\} \sin 3t \\
&- \dots \\
&- \left\{ \sum_{j=0}^2 \frac{(2m)_{j+n-3} (-n+1)_{j+n-3}}{(2m+n+1)_{j+n-3} (j+n-3)!} \frac{(2m)_j (-n)_j}{(2m+n+1)_j j!} \right\} \\
&\quad \cdot \sin(n-3)t \\
&- \left\{ \sum_{j=0}^1 \frac{(2m)_{j+n-2} (-n+1)_{j+n-2}}{(2m+n+1)_{j+n-2} (j+n-2)!} \frac{(2m)_j (-n)_j}{(2m+n+1)_j j!} \right\} \\
&\quad \cdot \sin(n-2)t \\
&- \frac{(2m)_{n-1} (-n+1)_{n-1}}{(2m+n+1)_{n-1} (n-1)!} \sin(n-1)t.
\end{aligned} \tag{14}$$

After all we obtain

$$\begin{aligned}
&d(t) \\
&= n \left[\left\{ \sum_{j=0}^{n-2} \frac{(2m)_{j+1} (-n+1)_{j+1}}{(2m+n+1)_j (j+1)!} \frac{(2m)_j (-n)_j}{(2m+n+1)_j j!} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\sum_{k=0}^{n-1} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+1}(-n)_{k+1}}{(2m+n+1)_{k+1}(k+1)!} \right) \sin t \\
& + \left\{ \left(\sum_{j=0}^{n-3} \frac{(2m)_{j+2}(-n+1)_{j+2}}{(2m+n+1)_k k!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!} \right) \right. \\
& - \left(\sum_{k=0}^{n-2} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+2}(-n)_{k+2}}{(2m+n+1)_{k+2}(k+2)!} \right) \sin 2t \\
& + \left\{ \left(\sum_{j=0}^{n-4} \frac{(2m)_{j+3}(-n+1)_{j+3}}{(2m+n+1)_{j+3}(j+3)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!} \right) \right. \\
& - \left(\sum_{k=0}^{n-3} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+3}(-n)_{k+3}}{(2m+n+1)_{k+3}(k+3)!} \right) \sin 3t \\
& + \dots \\
& + \left\{ \left(\sum_{j=0}^2 \frac{(2m)_{j+n-3}(-n+1)_{j+n-3}}{(2m+n+1)_{j+n-3}(j+n-3)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!} \right) \right. \\
& - \left(\sum_{k=0}^3 \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-3}(-n)_{k+n-3}}{(2m+n+1)_{k+n-3}(k+n-3)!} \right) \sin(n-3)t \\
& + \left\{ \left(\sum_{j=0}^1 \frac{(2m)_{j+n-2}(-n+1)_{j+n-2}}{(2m+n+1)_{j+n-2}(j+n-2)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!} \right) \right. \\
& - \left(\sum_{k=0}^2 \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-2}(-n)_{k+n-2}}{(2m+n+1)_{k+n-2}(k+n-2)!} \right) \sin(n-2)t \\
& + \left\{ \frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} \right. \\
& - \left(\sum_{k=0}^1 \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-1}(-n)_{k+n-1}}{(2m+n+1)_{k+n-1}(k+n-1)!} \right) \sin(n-1)t \\
& - \left. \frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} \sin nt \right\}. \tag{15}
\end{aligned}$$

From the above we obtain the following form,

$$\begin{aligned}
d(t) = n \{ & B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \dots + B_{n-2} \sin(n-2)t \\
& B_{n-1} \sin(n-1)t + B_n \sin nt \}. \tag{16}
\end{aligned}$$

We will determine the coefficients B_k , $k = 1, \dots, n$. In fact we have

$$B_n = - \frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} = (-1)^{n-1} \frac{(2m)_n}{(2m+n+1)_n}. \tag{17}$$

And we have

$$\begin{aligned}
B_{n-1} &= \frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} \\
&\quad - \sum_{k=0}^1 \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-1}(-n)_{k+n-1}}{(2m+n+1)_{k+n-1}(k+n-1)!} \\
&= \left\{ \frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} - \frac{(2m)_{n-1}(-n)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} \right\} \\
&\quad - \frac{(2m)(-n+1)}{(2m+n+1)1!} \frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} \\
&= (-1)^{n-1} \left\{ \frac{(2m)_{n-1}}{(2m+n+1)_{n-1}} + \frac{(2m)_{n-1}(-n)}{(2m+n+1)_{n-1}} \right\} \\
&\quad + \frac{(2m)}{(2m+n+1)} \frac{(2m)_n(n-1)}{(2m+n+1)_n} \\
&= (-1)^{n-2} \left\{ \frac{(2m)_{n-1}(n-1)}{(2m+n+1)_{n-1}} + \frac{(2m)}{(2m+n+1)} \frac{(2m)_n(n-1)}{(2m+n+1)_n} \right\} \\
&= (-1)^{n-2} \sum_{k=0}^1 \frac{(2m)_{k+n-1}(2m)_k}{(2m+n+1)_{k+n-1}(2m+n+1)_k} \\
&\quad \cdot \frac{(n-1)!n!(n-1)}{(n+k-1)!(1-k)!(n-k)!}. \tag{18}
\end{aligned}$$

For B_{n-r} , $r = 2, \dots, n-1$ we have

$$\begin{aligned}
B_{n-r} &= \sum_{j=0}^{r-1} \frac{(2m)_{j+n-r}(-n+1)_{j+n-r}}{(2m+n+1)_{j+n-r}(j+n-r)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!} \\
&\quad - \sum_{k=0}^r \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-r}(-n)_{k+n-r}}{(2m+n+1)_{k+n-r}(k+n-r)!} \\
&= \sum_{k=0}^{r-1} \frac{(2m)_{k+n-r}(-1)^{k+n-r}(n-1)!}{(2m+n+1)_{k+n-r}(k+n-r)!(r-k-1)!} \\
&\quad \cdot \frac{(2m)_k(-1)^k n!}{(2m+n+1)_k k!(n-k)!} \\
&\quad - \sum_{k=0}^{r-1} \frac{(2m)_k(-1)^k(n-1)!}{(2m+n+1)_k k!(n-k-1)!} \\
&\quad \cdot \frac{(2m)_{k+n-r}(-1)^{k+n-r} n!}{(2m+n+1)_{k+n-r}(k+n-r)!(r-k)!} \\
&\quad - \frac{(2m)_r(-1)^r(n-1)!}{(2m+n+1)_r r!(n-r-1)!} \frac{(2m)_n(-1)^n n!}{(2m+n+1)_n n!}. \tag{19}
\end{aligned}$$

And we see that

$$B_{n-r} = (-1)^{n-r} \sum_{k=0}^{r-1}$$

$$\begin{aligned}
& \frac{(2m)_{k+n-r}(2m)_k(n-1)!n!}{(2m+n+1)_{k+n-r}(2m+n+1)_k(k+n-r)!(r-1-k)!k!(n-k-1)!} \\
& \cdot \left\{ \frac{1}{n-k} - \frac{1}{r-k} \right\} \\
& + (-1)^{n-r-1} \frac{(2m)_r(2m)_n(n-1)!}{(2m+n+1)_r(2m+n+1)_n r!(n-r-1)!} \\
& = (-1)^{n-r-1} \sum_{k=0}^{r-1} \frac{(2m)_{k+n-r}(2m)_k n!(n-1)!(n-r)}{(2m+n+1)_{k+n-r}(2m+n+1)_k(k+n-r)!(r-k)!k!(n-k)!} \\
& + (-1)^{n-r-1} \frac{(2m)_n(2m)_r(n-1)!}{(2m+n+1)_n(2m+n+1)_r r!(n-r-1)!} \\
& = (-1)^{n-r-1} \sum_{k=0}^r \frac{(2m)_{k+n-r}(2m)_k(n-1)!n!(n-r)}{(2m+n+1)_{k+n-r}(2m+n+1)_k(k+n-r)!(r-k)!k!(n-k)!} \quad (20)
\end{aligned}$$

and in particular

$$\begin{aligned}
B_1 &= \sum_{k=0}^{n-1} \\
& \frac{(2m)_{k+1}(2m)_k(n-1)!n!}{(2m+n+1)_{k+1}(2m+n+1)_k(k+1)!(n-k-1)!k!(n-k)!}. \quad (21)
\end{aligned}$$

4. On Formula 3

By Theorem 1 it follows

$$\begin{aligned}
& |F(-n, 2m; 2m+n+1; e^{it})|^2 \\
&= \frac{(n+1)_n}{(2m+n+1)_n} \sum_{s=0}^n \frac{(-n)_s(n+1-s)_{n-s}(2m)_s y^s}{(-2n)_s(2m+n+1)_{n-s} s!} \\
&= \sum_{s=0}^n \frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! y^s,
\end{aligned}$$

where $y = 2(1 - \cos t)$. Therefore we see that

$$\begin{aligned}
& 2d(t) \\
&= \frac{d}{dt} |F(-n, 2m; 2m+n+1; e^{it})|^2 \\
&= \frac{(n+1)_n}{(2m+n+1)_n} \sum_{s=1}^n \frac{(-n)_s(n+1-s)_{n-s}(2m)_s s y^{s-1} 2 \sin t}{(-2n)_s(2m+n+1)_{n-s} s!} \\
&= \sum_{s=1}^n \frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! s y^{s-1} \\
& \quad \cdot 2 \sin t \quad (22)
\end{aligned}$$

and after all we obtain

$$\begin{aligned}
 & d(t) \\
 = & \sum_{s=1}^n \frac{(2m)_s}{(2m+n+1)_n (2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! s y^{s-1} \\
 & \cdot \sin t.
 \end{aligned} \tag{23}$$

From the above we can write $d(t)$ such as the following form,

$$\begin{aligned}
 d(t) = & n \sin t \left[A_0 + A_1 (2x)^2 + A_2 (2x)^4 + \cdots + A_{n-2} (2x)^{2(n-2)} \right. \\
 & \left. + A_{n-1} (2x)^{2(n-1)} \right],
 \end{aligned} \tag{24}$$

where let $x = \sin(t/2)$ and we see that

$$A_{n-r} = \frac{(2m)_{n-r+1}}{(2m+n+1)_n (2m+n+1)_{r-1}} \binom{n-1}{n-r} \binom{n+r-1}{n} (2(r-1))!. \tag{25}$$

5. On Formula 4

Substituting the following identity in the above (16) in place of $\sin(jt)$,

$$\sin(jt) = \cos \frac{t}{2} \sum_{k=0}^{j-1} (-1)^k \binom{j+k}{j-k-1} \left(2 \sin \frac{t}{2} \right)^{2k+1}, \tag{26}$$

then we see that

$$\begin{aligned}
 & d(t) \\
 = & n \sin t \left[B_1 + B_2 \frac{\sin 2t}{\sin t} + B_3 \frac{\sin 3t}{\sin t} + \cdots \right. \\
 & + B_{n-3} \frac{\sin(n-3)t}{\sin t} + B_{n-2} \frac{\sin(n-2)t}{\sin t} \\
 & + B_{n-1} \frac{\sin(n-1)t}{\sin t} + B_n \frac{\sin nt}{\sin t} \left. \right] \\
 = & n \sin t \left\{ B_1 \right. \\
 & + B_2 \sum_{k=0}^1 (-1)^k \binom{2+k}{2-k-1} \left(2 \sin \frac{t}{2} \right)^{2k} \\
 & + B_3 \sum_{k=0}^2 (-1)^k \binom{3+k}{3-k-1} \left(2 \sin \frac{t}{2} \right)^{2k} \\
 & + \cdots
 \end{aligned}$$

$$\begin{aligned}
& + B_{n-2} \sum_{k=0}^{n-3} (-1)^k \binom{n+k-2}{n-k-3} \left(2 \sin \frac{t}{2}\right)^{2k} \\
& + B_{n-1} \sum_{k=0}^{n-2} (-1)^k \binom{n+k-1}{n-k-2} \left(2 \sin \frac{t}{2}\right)^{2k} \\
& + B_n \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{n-k-1} \left(2 \sin \frac{t}{2}\right)^{2k} \}. \tag{27}
\end{aligned}$$

Let $x = \sin(t/2)$. Then from the above we can write $d(t)$ such as the following form,

$$\begin{aligned}
d(t) = n \sin t [& A_0 + A_1(2x)^2 + A_2(2x)^4 + \cdots + A_{n-2}(2x)^{2(n-2)} \\
& + A_{n-1}(2x)^{2(n-1)}]. \tag{28}
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
A_{n-r} & = B_n (-1)^{n-r} \binom{2n-r}{r-1} + B_{n-1} (-1)^{n-r} \binom{2n-r-1}{r-2} \\
& + B_{n-2} (-1)^{n-r} \binom{2n-r-2}{r-3} \\
& + B_{n-3} (-1)^{n-r} \binom{2n-r-3}{r-4} \\
& + \cdots \\
& + B_{n-r+2} (-1)^{n-r} \binom{2n-2r+2}{1} \\
& + B_{n-r+1} (-1)^{n-r} \binom{2(n-r)+1}{0} \\
& = \sum_{k=0}^{r-1} B_{n-k} (-1)^{n-r} \binom{2n-r-k}{r-k-1}. \tag{29}
\end{aligned}$$

From the above we see that the following combinatorial equality holds.

$$\begin{aligned}
& A_{n-r} \\
& = \frac{(2m)_{n-r+1}}{(2m+n+1)_n (2m+n+1)_{r-1}} \binom{n-1}{n-r} \binom{n+r-1}{n} (2(r-1))! \\
& = \sum_{k=0}^{r-1} \sum_{j=0}^k (-1)^{r+k+1} \frac{(2m)_{j+n-k} (2m)_j}{(2m+n+1)_{j+n-k} (2m+n+1)_j} \\
& \quad \cdot \frac{(n-1)! n! (n-k)}{(n+j-k)! (k-j)! j! (n-j)!} \binom{2n-r-k}{r-k-1}. \tag{30}
\end{aligned}$$

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